Spin 1/2 bosons etc. in a theory with Lorentz violation

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Abstract. An action with unconventional supersymmetry was introduced in an earlier paper. Here it is shown that this action leads to standard physics for fermions and gauge bosons at low energy, but to testable extensions of standard physics for fermions at high energy and for fundamental bosons which have not yet been observed. For example, the Lorentz-violating equation of motion for these bosons implies that they have spin 1/2.

1. Introduction

The following Euclidean action was postulated in an earlier paper [1]:

$$S=\int d^D x\left[\frac{1}{2m}\partial^M\Psi^\dagger\partial_M\Psi-\mu\Psi^\dagger\Psi+\frac{1}{2}b\left(\Psi^\dagger\Psi\right)^2\right] \tag{1.1}$$

with

$$\Psi=\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_N\end{pmatrix}\qquad,\qquad z=\begin{pmatrix} z_b\\ z_f\end{pmatrix}. \tag{1.2}$$

This action has "natural supersymmetry", in the sense that the initial bosonic fields z_b and fermionic fields z_f are treated in exactly the same way. The only difference is that the z_b are ordinary complex numbers whereas the z_f are anticommuting Grassmann numbers. (Here, as in Ref. 1, "supersymmetry" is taken to have its general definition [2, 3]: An action is supersymmetric if it is invariant under a transformation which converts fermions to bosons and vice-versa.) It was found in Ref. 1 that standard physics can emerge from (1.1) at energies that are far below the Planck scale, provided that specific kinds of topological defects are included in the theory. For example, one can obtain an $SO(10)$ grand-unified theory, containing both the Standard Model and a natural mechanism for small neutrino masses [4-15].

In the present paper, it will be shown that the theory predicts testable extensions of standard physics, both for fermions at high energy and for fundamental bosons which have not yet been observed.

2. Canonical Quantization in Lorentzian Spacetime

Path-integral quantization can ordinarily be replaced by canonical quantization, or vice-versa [16], through a procedure that is similar to that for a single particle. In the present theory, whether this can be done consistently is a nontrivial issue, because the resulting field theory has some very unconventional features. These will be discussed in Section 4, but in the present section it will simply be assumed that one can define quantized fields $\hat{\Psi}$ etc. in the usual way [16-24].

After a change from path-integral to canonical quantization, and an inverse Wick rotation from Euclidean to Lorentzian time (with $S_L = iS$), the action (1.1) becomes

$$\hat{S}_L = -\int d^D x \left[\frac{1}{2m} \eta^{MN} \partial_M \hat{\Psi}_L^\dagger \partial_N \hat{\Psi}_L - \mu \hat{\Psi}_L^\dagger \hat{\Psi}_L + \frac{1}{2} b \left(\hat{\Psi}_L^\dagger \hat{\Psi}_L \right)^2 \right] \tag{2.1}$$

where $\eta^{MN} = diag(-1, 1, ..., 1)$. This notation is rather awkward, however, so for the remainder of the paper we will let

$$\hat{S}_L \to S, \; \hat{\Psi}_L \to \Psi \tag{2.2}$$

with the understanding that these are now quantized operators in Lorentzian spacetime. It is also understood that raising and lowering of indices is now done with the Minkowski metric tensor:

$$A^\mu B_\mu = \eta^{\mu\nu} A_\mu B_\nu \quad \text{or in } D \text{ dimensions} \quad A^M B_M = \eta^{MN} A_M B_N. \tag{2.3}$$

Later in this paper we will introduce the metric tensor associated with gravity and general coordinate transformations. To avoid confusion, this metric tensor $g_{\mu\nu}$ will always be shown explicitly, and simple raising and lowering of indices will always have the interpretation (2.3).

With the above change of notation, and after an integration by parts, (2.1) becomes

$$S = -\int d^D x \left[-\frac{1}{2m} \Psi^\dagger \partial^M \partial_M \Psi - \mu \Psi^\dagger \Psi + \frac{1}{2} b \left(\Psi^\dagger \Psi \right)^2 \right]. \tag{2.4}$$

The resulting equation of motion is

$$\left[-\frac{1}{2m} \partial^M \partial_M - \mu + V_{vac} + b \Delta \left(\Psi^\dagger \Psi \right) \right] \Psi = 0 \quad , \quad V_{vac} = b \left\langle \Psi^\dagger \Psi \right\rangle_{vac} \tag{2.5}$$

where $\langle \cdots \rangle_{vac}$ represents a vacuum expectation value, and

$$\Psi^\dagger \Psi = \left\langle \Psi^\dagger \Psi \right\rangle_{vac} + \Delta \left(\Psi^\dagger \Psi \right). \tag{2.6}$$

For the remainder of this section, we will consider either the vacuum or a noninteracting free field in the vacuum. We then have

$$\left(-\frac{1}{2m} \partial^M \partial_M - \mu + V_{vac} \right) \Psi_b = 0 \quad , \quad \left(-\frac{1}{2m} \partial^M \partial_M - \mu + V_{vac} \right) \Psi_f = 0. \tag{2.7}$$

It will be assumed that the physical vacuum contains a condensate whose order parameter

$$\Psi_{cond} = \langle \Psi_b \rangle_{vac} \tag{2.8}$$

has the form

$$\Psi_{cond} = U \, n_{cond}^{1/2} \eta_0 \tag{2.9}$$

$$U^\dagger U = \eta_0^\dagger \eta_0 = 1. \tag{2.10}$$

(As discussed in the next section, Ψ_{cond} is dominantly due to a GUT field that condenses in the very early universe. In the present theory, it is not static, but instead exhibits rotations in space and time that are described by U.) It will also be assumed that the order parameter can be written in the form

$$\Psi_{cond} = \Psi_{ext} \left(x^\mu \right) \; \Psi_{int} \left(x^m, x^\mu \right) \tag{2.11}$$

$$\Psi_{ext} \left(x^\mu \right) = U_{ext} \left(x^\mu \right) \; n_{ext}^{1/2} \left(x^\mu \right) \eta_{ext} \tag{2.12}$$

$$\Psi_{int} = U_{int} \, n_{int}^{1/2} \eta_{int} \tag{2.13}$$

where η_{ext} and η_{int} are constant vectors, and the quantities in the lower equation can depend on x^μ as well as x^m. Let us define external and internal "superfluid velocities" by

$$mv_M = -iU^{-1}\partial_M U \tag{2.14}$$

or

$$mv_\mu = -iU_{ext}^{-1}\partial_\mu U_{ext} - iU_{int}^{-1}\partial_\mu U_{int} \tag{2.15}$$

$$mv_m = -iU_{int}^{-1}\partial_m U_{int}. \tag{2.16}$$

The fact that U is unitary implies that $\partial_M U^\dagger U = -U^\dagger \partial_M U$ with $U^\dagger = U^{-1}$, or

$$mv_M = i\partial_M U^\dagger U \tag{2.17}$$

so that

$$v_M^\dagger = v_M \,. \tag{2.18}$$

In this section we will assume that

$$\partial_\mu U_{int} = 0 \tag{2.19}$$

in which case there are separate equations of motion for external and internal spacetime:

$$\left(-\frac{1}{2m}\partial^\mu \partial_\mu - \mu_{ext}\right)\Psi_{ext} = 0 \tag{2.20}$$

$$\left(-\frac{1}{2m}\partial^m \partial_m - \mu_{int} + V_{vac}\right)\Psi_{int} = 0 \tag{2.21}$$

with $\mu_{int} = \mu - \mu_{ext}$. The quantities V_{vac}, μ_{int}, and Ψ_{int} are allowed to have a slow parametric dependence on x^μ, as long as $\partial^\mu \partial_\mu \Psi_{int}$ is negligible.

When (2.12), (2.15), and (2.19) are used in (2.20), we obtain

$$\eta_{ext}^\dagger n_{ext}^{1/2}\left[\left(\frac{1}{2}mv^\mu v_\mu - \frac{1}{2m}\partial^\mu \partial_\mu - \mu_{ext}\right) - i\left(\frac{1}{2}\partial^\mu v_\mu + v^\mu \partial_\mu\right)\right] n_{ext}^{1/2}\eta_{ext} = 0 \tag{2.22}$$

and its Hermitian conjugate

$$\eta_{ext}^\dagger n_{ext}^{1/2}\left[\left(\frac{1}{2}mv^\mu v_\mu - \frac{1}{2m}\partial^\mu \partial_\mu - \mu_{ext}\right) + i\left(\frac{1}{2}\partial^\mu v_\mu + v^\mu \partial_\mu\right)\right] n_{ext}^{1/2}\eta_{ext} = 0. \tag{2.23}$$

Subtraction gives the equation of continuity

$$\partial_\mu j_{ext}^\mu = 0 \quad , \quad j_{ext}^\mu = \eta_{ext}^\dagger \, n_{ext} v^\mu \eta_{ext} \tag{2.24}$$

and addition gives the Bernoulli equation

$$\frac{1}{2}m\bar{v}_{ext}^2 + P_{ext} = \mu_{ext} \tag{2.25}$$

where

$$\bar{v}_{ext}^2 = \eta_{ext}^\dagger \, v^\mu v_\mu \, \eta_{ext} \tag{2.26}$$

$$P_{ext} = -\frac{1}{2m} n_{ext}^{-1/2}\partial^\mu \partial_\mu n_{ext}^{1/2}. \tag{2.27}$$

In the present theory, the order parameter in external spacetime, Ψ_{ext}, has the symmetry group $U(1) \times SU(2)$. The "superfluid velocity" in external spacetime, v_μ, can then be written in terms of the identity matrix σ^0 and Pauli matrices σ^a :

$$v^\mu = v_\alpha^\mu \sigma^\alpha \quad , \quad \mu, \alpha = 0, 1, 2, 3. \tag{2.28}$$

It is assumed that the basic texture of the order parameter is such that

$$v_k^0 = v_0^a = 0 \;\;, \quad k, a = 1, 2, 3 \tag{2.29}$$

to a good approximation, yielding the simplification

$$\frac{1}{2} m v^{\alpha\mu} v_\mu^\alpha + P_{ext} = \mu_{ext}. \tag{2.30}$$

Let

$$\Delta\Psi_b = \Psi_b - \Psi_{cond} \tag{2.31}$$

and let Ψ_a represent either the bosonic field $\Delta\Psi_b$ or the fermionic field Ψ_f. If we start with the case of a free field, which interacts only with the condensate and other vacuum fields, (2.4) gives

$$S_a = -\int d^D x \, \Psi_a^\dagger \left(-\frac{1}{2m} \partial^M \partial_M - \mu + V_{vac} \right) \Psi_a. \tag{2.32}$$

Since Ψ_a satisfies a linear equation involving a Hermitian operator, it can be written in the form

$$\Psi_a \left(x^\mu, x^m \right) = \widetilde{\psi}_a^r \left(x^\mu \right) \psi_r^{int} \left(x^m \right) \tag{2.33}$$

with a summation implied over repeated indices, as usual. The $\widetilde{\psi}_a^r$ are field operators and the ψ_r^{int} are a complete set of basis functions in the internal space, which are required to be orthonormal,

$$\int d^{D-4} x \, \psi_r^{int\dagger} \left(x^m \right) \psi_{r'}^{int} \left(x^m \right) = \delta_{rr'} \, , \tag{2.34}$$

and to satisfy the internal equation of motion

$$\left(-\frac{1}{2m} \partial^m \partial_m - \mu_{int} + V_{vac} \right) \psi_r^{int} \left(x^m \right) = \varepsilon_r \psi_r^{int} \left(x^m \right). \tag{2.35}$$

(The ψ_r^{int} are allowed to have a slow parametric dependence on x^μ, as long as $\partial^\mu \partial_\mu \psi_r^{int}$ is negligible.) As usual, only the zero modes with $\varepsilon_r = 0$ will be kept, since the higher energies involve nodes in the internal space and are comparable to m_P. When (2.33)-(2.35) are used in (2.32), the result is

$$S_a = -\int d^4 x \, \widetilde{\psi}_a^\dagger \left(-\frac{1}{2m} \partial^\mu \partial_\mu - \mu_{ext} \right) \widetilde{\psi}_a \tag{2.36}$$

where $\widetilde{\psi}_a$ is the vector with components $\widetilde{\psi}_a^r$.

Let $\widetilde{\psi}_a$ be rewritten in the form

$$\widetilde{\psi}_a \left(x^\mu \right) = U_{ext} \left(x^\mu \right) \psi_a \left(x^\mu \right). \tag{2.37}$$

(The 2×2 matrix U_{ext} multiplies each of the 2-component operators $\widetilde{\psi}_a^r$.) Here ψ_a has a simple interpretation: It is the field seen by an observer in the frame of reference that is moving with the condensate. In the present theory, the GUT condensate Ψ_{cond} forms in the very early universe, and the other bosonic and fermionic fields Ψ_a are subsequently born into it. It is therefore natural to view them from the perspective of the condensate.

Equation (2.37) is, in fact, exactly analogous to rewriting the wavefunction of a particle in an ordinary superfluid moving with velocity v_s: $\psi_p' \left(x \right) = \exp \left(i v_s x \right) \psi_p \left(x \right)$. Here ψ_p and ψ_p' are the wavefunctions before and after a Galilean boost to the superfluid's frame of reference.

When (2.37) is substituted into (2.36), the result is

$$S_a = -\int d^4 x \; \psi_a^\dagger \left[\left(\frac{1}{2} m v^\mu v_\mu - \frac{1}{2m} \partial^\mu \partial_\mu - \mu_{ext} \right) - i \left(\frac{1}{2} \partial^\mu v_\mu + v^\mu \partial_\mu \right) \right] \psi_a. \tag{2.38}$$

If n_s and v_μ are slowly varying, so that P_{ext} and $\partial^\mu v_\mu$ can be neglected, (2.30) yields the simplification

$$S_a = \int d^4x\ \psi_a^\dagger \left(\frac{1}{2m} \partial^\mu \partial_\mu + i v_\alpha^\mu \sigma^\alpha \partial_\mu \right) \psi_a . \tag{2.39}$$

In the present theory, the gravitational vierbein is interpreted as the "superfluid velocity" associated with the GUT condensate Ψ_{cond}:

$$e_\alpha^\mu = v_\alpha^\mu . \tag{2.40}$$

Bosonic fields are conventionally represented as dimension 1 (rather than dimension $3/2$) operators, so let us define

$$\phi_b = \psi_b / \left(2m\right)^{1/2} . \tag{2.41}$$

Then the action for a free bosonic field is

$$S_b = \int d^4x\ \phi_b^\dagger \left(\partial^\mu \partial_\mu + 2mi e_\alpha^\mu \sigma^\alpha \partial_\mu \right) \phi_b \tag{2.42}$$

with

$$S_b \ \rightarrow \ \int d^4x\ \phi_b^\dagger \partial^\mu \partial_\mu \phi_b \quad \text{as } p_\mu \rightarrow \infty \tag{2.43}$$

for a plane-wave state $\phi_b \propto \exp\left(i p_\mu x^\mu\right)$. The usual form of the action for a massless and noninteracting bosonic field is thus regained at high energy.

For a free fermionic field, on the other hand, the action is

$$S_f = \int d^4x\ \psi_f^\dagger \left(\frac{1}{2m} \partial^\mu \partial_\mu + i e_\alpha^\mu \sigma^\alpha \partial_\mu \right) \psi_f \tag{2.44}$$

with

$$S_f \ \rightarrow \ \int d^4x\ \psi_f^\dagger \, i e_\alpha^\mu \sigma^\alpha \, \partial_\mu \psi_f \quad \text{as } p_\mu \rightarrow 0 \tag{2.45}$$

so the usual form of the action for a massless and noninteracting fermionic field is regained at low energy. To be more specific, the standard fermionic action is regained when

$$p^\mu \ll m v_\alpha^\mu \tag{2.46}$$

with $m \sim m_P$.

3. Origin of Gauge Fields

Let us now relax assumption (2.19) and allow U_{int} to vary with the external coordinates x^μ. It is convenient to write

$$\Psi_{int}\left(x^m\right) = \widetilde{U}_{int}\left(x^\mu, x^m\right) \bar{\Psi}_{int}\left(x^m\right) = \widetilde{U}_{int}\left(x^\mu, x^m\right) \bar{U}_{int}\ \left(x^m\right) n_{int}^{1/2}\left(x^m\right) \eta_{int} \tag{3.1}$$

where $n_{int}\left(x^m\right) = \bar{\Psi}_{int}^\dagger\left(x^m\right) \bar{\Psi}_{int}\left(x^m\right)$ and $\bar{\Psi}_{int}$ still satisfies the internal equation of motion

$$\left(-\frac{1}{2m} \partial^m \partial_m - \mu_{int} + V_{vac} \right) \bar{\Psi}_{int}\left(x^m\right) = 0. \tag{3.2}$$

This is a nonlinear equation because V_{vac} is largely determined by n_{int}.

The internal basis functions satisfy (2.35) with $\varepsilon_r = 0$:

$$\left(-\frac{1}{2m} \partial^m \partial_m - \mu_{int} + V_{vac} \right) \psi_r^{int}\left(x^m\right) = 0. \tag{3.3}$$

This is a linear equation because $V_{vac}\left(x^m\right)$ is now regarded as a known function.

If the vacuum of the internal space had a trivial topology, the solutions to (3.2) and (3.3) would be trivial, and the resulting universe would presumably not support nontrivial structures such as intelligent life. The full path integral involving (1.1) contains all configurations of the fields, however, including those with nontrivial topologies. In the present theory, the "geography" of the universe inhabited by human beings involves an internal instanton in

$$d = D - 4 \tag{3.4}$$

dimensions which is analogous to a $U(1)$ vortex in 2 dimensions or an $SU(2)$ instanton in 4 Euclidean dimensions. The standard features of four-dimensional physics – including gauge symmetries and chiral fermions – arise from the presence of this instanton.

In the following, it is not necessary to have a detailed knowledge of the internal instanton. The only property required is a d-dimensional spherical symmetry for the internal condensate, and, as a result, for the functions $\widetilde{\psi}_r^{int}$ defined by

$$\psi_r^{int} = \bar{U}_{int}\widetilde{\psi}_r^{int}. \tag{3.5}$$

To be specific, it is required that

$$K_i \, \widetilde{\psi}_r^{int} = 0 \tag{3.6}$$

where

$$K_i = K_i^n \partial_n \tag{3.7}$$

is a Killing vector associated with the spherical symmetry of the internal metric tensor g_{mn} defined below. At a given point, the derivatives of (3.7) involve only the $(d-1)$ angular coordinates, and not the radial coordinate r, so (3.6) states that n_{int} and the $\widetilde{\psi}_r^{int}$ are functions only of r.

Although a detailed description is not necessary, it is worthwhile to consider a concrete example, in which $V_{vac} = bn_{ext}n_{int} + V_0$ and V_0 is a constant. For clarity, we can start with a picture in which the instanton occupies an unbounded volume, and then move to a physically more acceptable description in which it is confined to a finite region $r < r_0$. The finite instanton has finite action, and can be viewed as a "spinning" ball of condensate. The corresponding order parameter has a node at $r = r_0$, from which the condensate rises to become fully formed at large r. The region $r < r_0$ corresponds to our physical universe, and the region $r > r_0$ is unobservable.

The same arguments that led to the external Bernoulli equation (2.25) also yield an internal Bernoulli equation

$$-\frac{1}{2m} n_{int}^{-1/2} \partial^m \partial_m n_{int}^{1/2} + \frac{1}{2} m \eta_{int}^{\dagger} v^m v_m \eta_{int} - \mu_{int} + V_{vac} = 0. \tag{3.8}$$

In our example, it is assumed that the instanton has the symmetry of a $(d-1)$-sphere, with

$$\eta_B^{\dagger} v^m v_m \eta_B = \left(\bar{a}/mr\right)^2 \tag{3.9}$$

$$\partial^m \partial_m n_{int}^{1/2} = \frac{1}{r^{d-1}} \frac{d}{dr} \left(r^{d-1} \frac{d}{dr} n_{int}^{1/2} \right). \tag{3.10}$$

Then (3.8) can be rewritten as

$$-\frac{1}{\rho^{d'}} \frac{d}{d\rho} \left(\rho^{d'} \frac{df}{d\rho} \right) + \frac{\bar{a}^2}{\rho^2} f + f^3 - f = 0 \tag{3.11}$$

where $d' = d - 1$, $\rho = r/\xi_{int}$, and $f = n_{int}^{1/2}/\bar{n}_{int}^{1/2}$, with $\xi_{int} = \left(2m\mu'_{int}\right)^{-1/2}$, $\mu'_{int} = \mu_{int} - V_0$, and $\bar{n}_{int} = \mu'_{int}/bn_{ext}$. The asymptotic solutions to (3.11) are

$$f \propto \rho^n \quad \text{as } \rho \to 0 \tag{3.12}$$

$$f = 1 - \bar{a}^2/2\rho^2 \quad \text{as } \rho \to \infty \tag{3.13}$$

where

$$n = \frac{1}{2}\left[\sqrt{(d-2)^2 + 4\bar{a}^2} - (d-2)\right] \tag{3.14}$$

so that

$$n = 1 \quad \text{if } \bar{a}^2 = d - 1. \tag{3.15}$$

It is easy to show that (3.15) holds for a minimal vortex in two dimensions or a minimal $SU(2)$ instanton in four dimensions.

Since the volume element is proportional to $\rho^{d-1}d\rho$ and $1 - f^2$ is proportional to ρ^{-2} as $\rho \to \infty$, the above solution has infinite action. However, we can obtain a solution with finite action by requiring that

$$\Psi_{int} = R\,(r)\;\bar{n}_{int}^{1/2}\,U_{int}\,\eta_{int} \quad , \quad \rho < \rho_0 \tag{3.16}$$

$$\Psi_{int} = 0 \quad , \quad \rho = \rho_0 \tag{3.17}$$

$$\Psi_{int} = \bar{R}\,(r)\;\eta_{int} \quad , \quad \rho > \rho_0 \tag{3.18}$$

so that the instanton is confined to the region inside a radius ρ_0 which is determined by the boundary conditions below. Then (3.11) is replaced by

$$-\frac{1}{\rho^{d'}}\frac{d}{d\rho}\left(\rho^{d'}\frac{dR}{d\rho}\right) + \frac{\bar{a}^2}{\rho^2}R + R^3 - R \;\; = 0 \quad , \quad \rho < \rho_0 \tag{3.19}$$

$$-\frac{1}{2m}\frac{1}{r^{d'}}\frac{d}{dr}\left(r^{d'}\frac{d\bar{R}}{dr}\right) + bn_{ext}\bar{R}^3 - \mu\bar{R} = 0 \quad , \quad \rho > \rho_0. \tag{3.20}$$

R is required to satisfy (3.19) with the boundary condition $R \to 0+$ as $\rho \to 0$. $\bar{R}$ is required to satisfy (3.20) with the boundary condition $\bar{R} \to -\left(\mu/bn_{ext}\right)^{1/2}$ as $r \to \infty$ (and with $\partial\Psi_{int}/\partial r$ continuous at $\rho = \rho_0$). In the following, we will be concerned only with the physical region $\rho < \rho_0$, and the integrals are over only this region; e.g.,

$$V_{int} = \int d^d x = \int_{\rho<\rho_0} d^d x. \tag{3.21}$$

The above treatment assumes that the second-order equations (3.19) and (3.20) are exact. However, in a picture that will be presented elsewhere [25], the continuum approximation is not perfect, and as a result higher derivatives can be significant near the Planck scale. For an nth order differential equation, we have the freedom to impose n boundary conditions. This fact makes it possible to satisfy (3.16)-(3.17) for various values of ρ_0, so that the volume V_{int} of the internal space is largely arbitrary. As in other Kaluza-Klein theories, V_{int} determines the strength of gravitational and gauge interactions, so the arbitrariness of V_{int} has obvious anthropic implications.

The vierbein e_α^μ of external spacetime was defined in (2.40). It is convenient to define the remaining components of the vielbein in a slightly different way, by representing mv_M in terms of a set of matrices σ^A,

$$v_M = v_{MA}\sigma^A = v_{M\alpha}\sigma^\alpha + v_{Mc}\sigma^c, \tag{3.22}$$

and letting

$$e_{Mc} = -v_{Mc} \quad , \quad M = 0, 1, ..., D-1 \quad , \quad c \geq 4. \tag{3.23}$$

(The σ^α are associated with U_{ext}, and the σ^c with U_{int}. Since (2.16) implies that $v_{m\alpha} = 0$, all the nonzero e_{MA} have now been specified.) When (2.19) holds, the only nonzero components of the metric tensor are

$$g^{\mu\nu} = \eta^{\alpha\beta}e_\alpha^\mu e_\beta^\nu. \tag{3.24}$$

and

$$g_{mn} = e_{mc} e_{nc} \tag{3.25}$$

which are respectively associated with external spacetime and the internal space. More generally, however, mv_μ contains a contribution

$$mv_{\mu c}\sigma^c = -i\widetilde{U}_{int}^{-1}\left(x^\mu, x^m\right)\partial_\mu \widetilde{U}_{int}\left(x^\mu, x^m\right) \tag{3.26}$$

so that $e_{\mu c}$ is nonzero and the metric tensor has off-diagonal components

$$g_{\mu m} = e_{\mu c} e_{mc}. \tag{3.27}$$

In the present theory, just as in classic Kaluza-Klein theories, it is appropriate to write

$$e_{\mu c} = A_\mu^i K_i^n v_{nc} \quad , \quad g_{\mu m} = A_\mu^i K_i^n g_{mn} \tag{3.28}$$

or, for later convenience,

$$mv_{\mu c}\sigma^c = -A_\mu^i \sigma_i \tag{3.29}$$

$$\sigma_i = mK_i^n v_{nc}\sigma^c. \tag{3.30}$$

For simplicity of notation, let

$$\langle r|Q|s\rangle = \int d^d x \, \psi_r^{int\dagger} Q \psi_s^{int} \quad \text{with} \quad \langle r|s\rangle = \delta_{rs} \tag{3.31}$$

for any operator Q, so that (3.5)-(3.7) and (2.16) give

$$\langle r|\left(-iK_i\right)|s\rangle = \langle r|\left(-iK_i^n\right)\left(imv_n\right)|s\rangle = \langle r|\sigma_i|s\rangle \, . \tag{3.32}$$

With the definition

$$t_i^{rs} = \langle r|\left(-iK_i\right)|s\rangle \tag{3.33}$$

we then have

$$\langle r|\sigma_i|s\rangle = t_i^{rs}. \tag{3.34}$$

The Killing vectors have an algebra

$$K_i K_j - K_j K_i = -c_{ij}^k K_k \tag{3.35}$$

or

$$\left(-iK_i\right)\left(-iK_j\right) - \left(-iK_j\right)\left(-iK_i\right) = ic_{ij}^k\left(-iK_k\right) \tag{3.36}$$

so the same is true of the matrices t_i^{rs}:

$$t_i t_j - t_j t_i = ic_{ij}^k t_k. \tag{3.37}$$

With the more general version of (2.33) and (2.37),

$$\Psi_a\left(x^\mu, x^m\right) = U_{ext}\left(x^\mu\right)\widetilde{U}_{int}\left(x^\mu, x^m\right)\psi_a^r\left(x^\mu\right)\psi_r^{int}\left(x^m\right), \tag{3.38}$$

we have

$$\partial_\mu \Psi_a = U_{ext}\left(x^\mu\right)\widetilde{U}_{int}\left(x^\mu, x^m\right)\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha + imv_{\mu c}\sigma^c\right)\psi_a^r \psi_r^{int} \tag{3.39}$$

and

$$\begin{aligned}
&\int d^d x \, \Psi_a^\dagger \, \partial^\mu \partial_\mu \, \Psi_a \\
&= \int d^d x \, \psi_r^{int\dagger}\psi_a^{r\dagger}\eta^{\mu\nu}\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha + imv_{\mu c}\sigma^c\right)\left(\partial_\nu + imv_{\nu\beta}\sigma^\beta + imv_{\nu d}\sigma^d\right)\psi_a^s \psi_s^{int} \\
&= \psi_a^{r\dagger}\,\eta^{\mu\nu}\langle r|\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha + imv_{\mu c}\sigma^c\right)\sum_t |t\rangle\langle t| \left(\partial_\nu + imv_{\nu\beta}\sigma^\beta + imv_{\nu d}\sigma^d\right)|s\rangle \, \psi_a^s \\
&= \psi_a^{r\dagger}\,\eta^{\mu\nu}\left[\delta_{rt}\left(\partial_\mu + imv_{\mu\alpha}\sigma^\alpha\right) - iA_\mu^i t_i^{rt}\right]\left[\delta_{ts}\left(\partial_\nu + imv_{\nu\beta}\sigma^\beta\right) - iA_\nu^j t_j^{ts}\right]\psi_a^s \\
&= \psi_a^\dagger\,\eta^{\mu\nu}\left[\left(\partial_\mu - iA_\mu^i t_i\right) + imv_{\mu\alpha}\sigma^\alpha\right]\left[\left(\partial_\nu - iA_\nu^j t_j\right) + imv_{\nu\beta}\sigma^\beta\right]\psi_a
\end{aligned} \tag{3.40}$$

where (2.34), (3.29), and (3.34) have been used. The action (2.32) then becomes

$$S_a = \int d^4x\, \psi_a{}^\dagger \left(\frac{1}{2m} D^\mu D_\mu + \frac{1}{2} i v^\mu_\alpha \sigma^\alpha D_\mu + \frac{1}{2} D_\mu i v^\mu_\alpha \sigma^\alpha - \frac{1}{2} m v^{\alpha\mu} v^\alpha_\mu + \mu_{ext} \right) \psi_a \qquad (3.41)$$

after (2.35) is used, where

$$D_\mu = \partial_\mu - i A^i_\mu t_i . \qquad (3.42)$$

With the approximations above (2.39), (2.30) and (2.40) imply that

$$S_a = \int d^4x\, \psi_a{}^\dagger \left(\frac{1}{2m} D^\mu D_\mu + i e^\mu_\alpha \sigma^\alpha D_\mu \right) \psi_a . \qquad (3.43)$$

This is in fact the generalization of (2.39) when the "internal order parameter" is permitted to vary as a function of the external coordinates x^μ.

As in Ref. 1, let us postulate a cosmological model in which

$$e^\mu_\alpha = \lambda \delta^\mu_\alpha \equiv \widetilde{e}^\mu_\alpha . \qquad (3.44)$$

In this case (3.43) can be rewritten as

$$S_a = \int d^4x\, \widetilde{g}\, \bar{\psi}_a{}^\dagger \left(\bar{m}^{-1} \widetilde{g}^{\mu\nu} D_\mu D_\nu + i e^\mu_\alpha \sigma^\alpha D_\mu \right) \bar{\psi}_a \qquad (3.45)$$

where

$$\widetilde{g}^{\mu\nu} \equiv \eta^{\alpha\beta} \widetilde{e}^\mu_\alpha \widetilde{e}^\nu_\beta \quad , \quad \bar{m} = 2\lambda^2 m \qquad (3.46)$$

$$\widetilde{g} \;\; = \left(-\det \widetilde{g}_{\mu\nu} \right)^{1/2} = \lambda^{-4} \quad , \quad \bar{\psi}_a = \lambda^2 \psi_a . \qquad (3.47)$$

(The tilde is a reminder that the above form is not general, and that $\widetilde{g}^{\mu\nu}$ is not a dynamical quantity.) In a locally inertial coordinate system with $e^\mu_\alpha = \delta^\mu_\alpha$, this becomes

$$S_a = \int d^4x\;\, \psi_a{}^\dagger \left(\bar{m}^{-1} \eta^{\mu\nu} D_\mu D_\nu + i \sigma^\mu D_\mu \right) \psi_a \qquad (3.48)$$

where the bar has been removed from ψ_a for simplicity, so the fermionic and bosonic actions are respectively

$$S_f = \int d^4x\;\, \psi_f{}^\dagger \left(\bar{m}^{-1} \eta^{\mu\nu} D_\mu D_\nu + i \sigma^\mu D_\mu \right) \psi_f \qquad (3.49)$$

and

$$S_b = \int d^4x\, \phi_b{}^\dagger \left(\eta^{\mu\nu} D_\mu D_\nu + i \bar{m} \sigma^\mu D_\mu \right) \phi_b \qquad (3.50)$$

where now

$$\phi_b = \psi_b / \bar{m}^{1/2} . \qquad (3.51)$$

Again, one regains the usual bosonic action at high energy,

$$S_b \;\rightarrow\; \int d^4x\, \phi_b^\dagger \eta^{\mu\nu} D_\mu D_\nu \phi_b \quad \text{for } p^\mu \gg \bar{m}, \qquad (3.52)$$

and the usual fermionic action at low energy,

$$S_f \;\rightarrow\; \int d^4x\, \psi_f^\dagger \, i\sigma^\mu \, D_\mu \psi_f \quad \text{for } p^\mu \ll \bar{m}, \qquad (3.53)$$

where the expressions now include gauge couplings and are written in a locally inertial coordinate system.

Recall that the initial gauge group is the same as the group of rotations in the internal space – e.g., $SO(10)$ for $d = 10$. The generators t_i correspond to a reducible representation of this group, composed of some set of irreducible representations that are left unspecified in the present paper, although it is clear that one can place the three generations of Standard Model fermions in three spinorial **16** representations. Each field will necessarily have a superpartner with the same quantum numbers, just as in standard supersymmetry [26, 27]. We leave the phenomenology of these fields for future work.

Notice that the deviations from standard physics in (3.49) and (3.50) are predicted only for (i) fermions at very high energy and (ii) fundamental bosons which have not yet been observed. Notice also that the present theory preserves both gauge invariance and many features of Lorentz invariance, including rotational invariance and the requirement that all massless particles travel with the same speed $c = 1$ in a locally inertial coordinate system. (This last feature follows from (4.18)-(4.21).) It appears that the present theory is in agreement with even the most sensitive tests of Lorentz invariance that are currently available [28]. Furthermore, issues like causality and logical consistency can ultimately be resolved by returning to the original action (1.1).

4. Consistency of Canonical Quantization

Let us now consider whether the present theory permits a consistent extension of standard field theory [16-24]. This is not a trivial issue because, as mentioned above, the fermionic action (3.49) is Lorentz invariant only at low energy ($p^\mu \ll \bar{m}$), and the bosonic action (3.50) has its usual form only at high energy ($p^\mu \gg \bar{m}$).

We will, in fact, encounter a difficulty which is essentially the same as that encountered in covariant quantization of the electromagnetic field [18]. Let $\zeta_{p\lambda}$ be the norm of a one-particle state $|1_{p\lambda}\rangle$:

$$\langle 1_{p\lambda}|1_{p\lambda}\rangle = \zeta_{p\lambda} \quad , \quad |1_{p\lambda}\rangle = a^\dagger_{p\lambda}|0\rangle \tag{4.1}$$

where p is the momentum and λ is the index defined below in (4.24). As in the case of the electromagnetic field, the quantization condition (4.30) or (4.48) will imply that there are intermediate states with negative norm. Even though such states can be consistently treated with the formalism of Gupta and Bleuler [18, 29, 30], they are not physical, so it is necessary to require that

$$a_{p\lambda}\,|phys\rangle\ = 0 \quad \text{if} \quad \omega_{p\lambda} > 0 \quad \text{and} \quad \zeta_{p\lambda} < 0 \tag{4.2}$$

$$b_{p\lambda}\,|phys\rangle\ = 0 \quad \text{if} \quad \omega_{p\lambda} < 0 \quad \text{and} \quad \zeta_{p\lambda} < 0 \tag{4.3}$$

where $a_{p\lambda}$ and $b_{p\lambda}$ are the particle and antiparticle destruction operators introduced below, and $|phys\rangle$ represents any physical state. (For the electromagnetic field, one can choose a gauge such that this condition is satisfied separately for all unphysical photons, although in a general Lorentz gauge it can be relaxed to $(a_{p3} - a_{p0})\,|phys\rangle = 0$, because the contributions of longitudinal and scalar photons then cancel.) For fermions in the present context, the condition for a single-particle state with positive norm will turn out to be

$$1 + 2\omega_n/\bar{m} > 0. \tag{4.4}$$

(See (4.33).) According to (4.18)-(4.21), there is always one function of the form $u_p \exp i\vec{p}\cdot\vec{x}$ satisfying this requirement, with

$$\lambda = 1 \tag{4.5}$$

where u_p is the right-handed 2-component spinor defined below in (4.22). There is also one positive-norm function of the form $v_p \exp i\vec{p}\cdot\vec{x}$, with

$$\lambda = 3 \ \text{ for } \ |\vec{p}| < \bar{m}/2 \quad , \quad \lambda = 4 \ \text{ for } \ |\vec{p}| > \bar{m}/2 \tag{4.6}$$

where v_p is the left-handed 2-component spinor defined below in (4.23). For bosons, on the other hand, the states with positive norm must satisfy

$$1 + \bar{m}/2\omega_n > 0. \tag{4.7}$$

(See (4.50).) There are always two functions of the form $u_p \exp i\vec{p}\cdot\vec{x}$ satisfying this condition, with

$$\lambda = 1 \text{ or } 2. \tag{4.8}$$

There is also at least one positive-norm function of the form $v_p \exp i\vec{p}\cdot\vec{x}$, with

$$\lambda = 4 \text{ for } |\vec{p}| < \bar{m}/2, \quad \lambda = 3 \text{ for } \bar{m}/2 < |\vec{p}| < \bar{m}, \quad \lambda = 3 \text{ or } 4 \text{ for } |\vec{p}| > \bar{m}. \tag{4.9}$$

In summary, $\zeta_{p\lambda}$ is > 0 for one-particle states satisfying (4.4)-(4.6) for fermions and (4.7)-(4.9) for bosons, with $\zeta_{p\lambda} < 0$ otherwise.

Let ψ and ϕ represent 2-component, complex fermionic and bosonic fields. In the case of fermions, and with gauge fields omitted, (3.49) gives

$$\mathcal{L}_\psi = -\,\bar{m}^{-1}\eta^{\mu\nu}\partial_\mu\psi^\dagger\partial_\nu\psi + \frac{1}{2}\left(i\psi^\dagger\sigma^\mu\partial_\mu\psi + h.c.\right) \tag{4.10}$$

$$= \bar{m}^{-1}\left(\dot{\psi}^\dagger\dot{\psi} - \partial^k\psi^\dagger\partial_k\psi\right) + \frac{1}{2}\left(i\psi^\dagger\dot{\psi} + i\psi^\dagger\sigma^k\partial_k\psi + h.c.\right) \tag{4.11}$$

where $\dot{\psi} = \partial_0\psi$ and "$h.c.$" means "Hermitian conjugate". The canonical momenta are (in a convenient but slightly unconventional notation)

$$\pi_\psi^\dagger = \frac{\partial\mathcal{L}_\psi}{\partial\dot{\psi}} = \bar{m}^{-1}\dot{\psi}^\dagger + \frac{1}{2}i\psi^\dagger \tag{4.12}$$

$$\pi_\psi = \frac{\partial\mathcal{L}_\psi}{\partial\dot{\psi}^\dagger} = \bar{m}^{-1}\dot{\psi} - \frac{1}{2}i\psi \tag{4.13}$$

and the Hamiltonian density is

$$\mathcal{H}_\psi = \pi_\psi^\dagger\dot{\psi} + \dot{\psi}^\dagger\pi_\psi - \mathcal{L}_\psi \tag{4.14}$$

$$= \bar{m}^{-1}\left(\dot{\psi}^\dagger\dot{\psi} + \partial^k\psi^\dagger\partial_k\psi\right) - \frac{1}{2}\left(i\psi^\dagger\sigma^k\partial_k\psi + h.c.\right). \tag{4.15}$$

From (4.10) we obtain the equation of motion

$$\bar{m}^{-1}\,\eta^{\mu\nu}\partial_\mu\partial_\nu\psi + i\sigma^\mu\partial_\mu\psi = 0. \tag{4.16}$$

Let $a_n\psi_n$ be a solution to this equation. Then we can write

$$\psi = \sum_n a_n\psi_n. \tag{4.17}$$

For each 3-momentum $\vec{p}$, there are four solutions to (4.16):

$$\psi_{p1} = A_{p1}\, u_p\, e^{i\vec{p}\cdot\vec{x}} \quad , \; a_{p1} = e^{-i\omega_{p1}x^0}a_{p1}(0) \quad , \; \omega_{p1} = |\vec{p}| \tag{4.18}$$

$$\psi_{p2} = A_{p2}\, u_p\, e^{i\vec{p}\cdot\vec{x}} \quad , \; a_{p2} = e^{-i\omega_{p2}x^0}a_{p2}(0) \quad , \; \omega_{p2} = -\bar{m} - |\vec{p}| \tag{4.19}$$

$$\psi_{p3} = A_{p3}\, v_p\, e^{i\vec{p}\cdot\vec{x}} \quad , \; a_{p3} = e^{-i\omega_{p3}x^0}a_{p3}(0) \quad , \; \omega_{p3} = -|\vec{p}| \tag{4.20}$$

$$\psi_{p4} = A_{p4}\, v_p\, e^{i\vec{p}\cdot\vec{x}} \quad , \; a_{p4} = e^{-i\omega_{p4}x^0}a_{p4}(0) \quad , \; \omega_{p4} = -\bar{m} + |\vec{p}| \tag{4.21}$$

where

$$\vec{\sigma}\cdot\vec{p}\,u_p = +\,|\vec{p}|\,u_p \tag{4.22}$$

$$\vec{\sigma}\cdot\vec{p}\,v_p = -\,|\vec{p}|\,v_p \tag{4.23}$$

$$n \leftrightarrow \vec{p},\lambda \quad \text{with } \lambda = 1,2,3,4 \tag{4.24}$$

and the $A_{p\lambda}$ are normalization constants specified below. We can choose

$$u_p^\dagger u_p = v_p^\dagger v_p = 1 \quad , \quad u_p^\dagger v_p = v_p^\dagger u_p = 0 \tag{4.25}$$

$$u_p u_p^\dagger + v_p v_p^\dagger = \mathbf{1} \tag{4.26}$$

where $\mathbf{1}$ is the 2×2 identity matrix. Since

$$\dot{a}_n = -i\omega_n a_n \quad , \quad \dot{a}_n^\dagger = i\omega_n a_n^\dagger \tag{4.27}$$

(4.12) and (4.13) give

$$\pi_\psi^\dagger = \frac{1}{2} i \sum_n \left(1 + 2\omega_n/\bar{m}\right) a_n^\dagger \psi_n^\dagger \tag{4.28}$$

$$\pi_\psi = -\frac{1}{2} i \sum_n \left(1 + 2\omega_n/\bar{m}\right) a_n \psi_n . \tag{4.29}$$

We quantize by interpreting ψ and $\pi^\dagger$ as operators, and requiring that

$$\left[\psi\left(\vec{x}, x^0\right), \pi_\psi^\dagger\left(\vec{x}\,', x^0\right)\right]_+ = i\delta\left(\vec{x} - \vec{x}\,'\right)\mathbf{1} \tag{4.30}$$

or more explicitly

$$\left[\psi_\alpha\left(\vec{x}, x^0\right), \pi_{\psi\beta}^\dagger\left(\vec{x}\,', x^0\right)\right]_+ = i\delta\left(\vec{x} - \vec{x}\,'\right)\delta_{\alpha\beta} \tag{4.31}$$

where α and β label the two components of ψ and $\pi_\psi^\dagger$, with $[X, Y]_\pm = XY \pm YX$. This requirement will be satisfied if

$$\left[a_n, a_m^\dagger\right]_+ = \delta_{nm}\, \zeta_n^f \tag{4.32}$$

$$\zeta_n^f = sgn\left(1 + 2\omega_n/\bar{m}\right) \tag{4.33}$$

$$\left[a_n, a_m\right]_+ = \left[a_n^\dagger, a_m^\dagger\right]_+ = 0 \tag{4.34}$$

$$A_n^* A_n = V^{-1}\left|1 + 2\omega_n/\bar{m}\right|^{-1} \tag{4.35}$$

where V is the normalization volume, since

$$\begin{aligned}
\frac{1}{2}\sum_n \left|1 + 2\omega_n/\bar{m}\right| \psi_n\left(\vec{x}\right)\psi_n^\dagger\left(\vec{x}\,'\right) &= \frac{1}{2}\sum_{\vec{p}\ \lambda=1,2} \left|1 + 2\omega_{p\lambda}/\bar{m}\right| A_{p\lambda}A_{p\lambda}^* u_p u_p^\dagger e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')} \\
&\quad + \frac{1}{2}\sum_{\vec{p}\ \lambda=3,4} \left|1 + 2\omega_{p\lambda}/\bar{m}\right| A_{p\lambda}A_{p\lambda}^* v_p v_p^\dagger e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')} \\
&= V^{-1}\sum_{\vec{p}} e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')}\left(u_p u_p^\dagger + v_p v_p^\dagger\right) \qquad (4.36) \\
&= \delta\left(\vec{x} - \vec{x}\,'\right)\mathbf{1}. \qquad (4.37)
\end{aligned}$$

The equation of motion (4.16) implies that $\mathcal{L}_\psi$ has the form $\partial_\mu \mathcal{M}^\mu$, so the Hamiltonian density of (4.14) is

$$\mathcal{H}_\psi = \pi_\psi^\dagger \dot{\psi} + \dot{\psi}^\dagger \pi_\psi - \partial_\mu \mathcal{M}^\mu . \tag{4.38}$$

Suppose for the moment that we are interested in the time-averaged value $\bar{H}$ of the Hamiltonian. In this context the last term can be ignored, since it does not contribute to

the four-dimensional integral when boundary terms are ignored. In any state, we then have

$$\left\langle \bar{H}_\psi \right\rangle = \int d^3x \left\langle \bar{\mathcal{H}}_\psi \right\rangle \tag{4.39}$$

$$= \sum_n \omega_n \left(1 + 2\omega_n/\bar{m}\right) \left\langle a_n^\dagger a_n \right\rangle \int d^3x \, \psi_n^\dagger \psi_n \tag{4.40}$$

$$= \sum_{n,\omega_n>0} \left\langle a_n^\dagger a_n \, \zeta_n^f \right\rangle |\omega_n| - \sum_{n,\omega_n<0} \left\langle b_n b_n^\dagger \, \zeta_n^f \right\rangle |\omega_n| \tag{4.41}$$

$$= \sum_n \left\langle N_n^f \right\rangle |\omega_n| - \sum_{n,\omega_n<0} |\omega_n| \tag{4.42}$$

where

$$N_n^f = a_n^\dagger a_n \, \zeta_n^f \ , \ \omega_n > 0 \tag{4.43}$$

$$N_n^f = b_n^\dagger b_n \, \zeta_n^f \ , \ \omega_n < 0 \tag{4.44}$$

$$b_n^\dagger = a_n \quad , \ b_n = a_n^\dagger \tag{4.45}$$

$$\left[b_n, b_m^\dagger\right]_+ = \left[a_n, a_m^\dagger\right]_+ = \delta_{nm} \, \zeta_n^f. \tag{4.46}$$

The above treatment can be repeated for the fundamental bosons described by (3.50), with

$$\psi \to \phi, \ a_n \to c_n, \ A_n \to B_n, \ b_n \to d_n \tag{4.47}$$

$$\left[\phi\left(\vec{x}, x^0\right), \pi_\phi^\dagger\left(\vec{x}\,', x^0\right)\right]_- = i\delta\left(\vec{x} - \vec{x}\,'\right) \mathbf{1} \tag{4.48}$$

$$\left[c_n, c_m^\dagger\right]_- = \delta_{nm} \, \zeta_n^b \, \omega_n / |\omega_n| \tag{4.49}$$

$$\zeta_n^b = sgn\left(1 + \bar{m}/2\omega_n\right) \tag{4.50}$$

$$\left[c_n, c_m\right]_- = \left[c_n^\dagger, c_m^\dagger\right]_- = 0 \tag{4.51}$$

$$\phi_n^\dagger\left(\vec{x}\right) \phi_n\left(\vec{x}\right) = B_n^* B_n = \left(2\,|\omega_n|\, V\right)^{-1} |1 + \bar{m}/2\omega_n|^{-1} \tag{4.52}$$

$$N_n^b = c_n^\dagger c_n \, \zeta_n^b \ , \ \omega_n > 0 \tag{4.53}$$

$$N_n^b = d_n^\dagger d_n \, \zeta_n^b \ , \ \omega_n < 0 \tag{4.54}$$

$$d_n^\dagger = c_n \quad , \ d_n = c_n^\dagger \tag{4.55}$$

$$\left[d_n, d_m^\dagger\right]_- = -\left[c_n, c_m^\dagger\right]_- = \delta_{nm} \, \zeta_n^b \quad , \ \omega_n < 0. \tag{4.56}$$

$$\left\langle \bar{H}_\phi \right\rangle = \sum_{n,\omega_n>0} \left\langle c_n^\dagger c_n \, \zeta_n^b \right\rangle |\omega_n| + \sum_{n,\omega_n<0} \left\langle d_n d_n^\dagger \, \zeta_n^b \right\rangle |\omega_n| \tag{4.57}$$

$$= \sum_n \left\langle N_n^b \right\rangle |\omega_n| + \sum_{n,\omega_n<0} |\omega_n| \,. \tag{4.58}$$

Returning to (4.15), one can use (4.18)-(4.23) to obtain the operator H itself:

$$H = H_\psi + H_\phi \tag{4.59}$$

$$H_\psi = \sum_n N_n^f \, |\omega_n| - \sum_{n,\omega_n<0} |\omega_n| \tag{4.60}$$

$$H_\phi = \sum_n N_n^b \, |\omega_n| + \sum_{n,\omega_n<0} |\omega_n| \,. \tag{4.61}$$

The total energy in any state is then

$$\langle H \rangle = \sum_n \left\langle N_n^f \right\rangle |\omega_n| + \sum_n \left\langle N_n^b \right\rangle |\omega_n| \,. \tag{4.62}$$

In particular, there is a cancellation of the bosonic and fermionic contributions to the vacuum energy (before the initial supersymmetry of the present theory is broken) just as in standard supersymmetry [27]:

$$\langle 0 \,| H |\, 0 \rangle = 0. \tag{4.63}$$

These results are not as trivial as they may seem, because the Lagrangians of (3.49) and (3.50) violate Lorentz invariance, and the ω_n are given by (4.18)-(4.21).

Other conserved quantities will also have their usual forms, because they also involve products like $\pi_\psi^\dagger \partial_\nu \psi$ or $\pi_\psi^\dagger \Delta\psi$: According to Noether's theorem for a single field χ, a conserved current has the form [17, 18]

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \chi\right)} \Delta\chi + a^\nu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \chi\right)} \partial_\nu \chi - \mathcal{L} \delta^\mu{}_\nu \right] \tag{4.64}$$

so that

$$j^0 = \pi_\chi^\dagger \Delta\chi + a^\nu \left(\pi_\chi^\dagger \partial_\nu \chi - \mathcal{L} \delta^0{}_\nu \right) \tag{4.65}$$

where each a^ν represents an independent shift of coordinates and $\Delta\chi$ represents the effect of a rotation or gauge transformation. Using the corresponding result for the two fields ψ and $\psi^\dagger$, or ϕ and $\phi^\dagger$, one obtains the usual expressions for the momenta, angular momenta, and charges.

The most dramatic prediction of this paper is that the bosons described by ϕ have an equation of motion which requires them to transform as spin 1/2 particles. Furthermore, there is a clear breaking of particle-antiparticle symmetry in the representation of ϕ even at low energy. Both of these features are associated with the fact that the present theory violates Lorentz invariance. The inapplicability of the usual spin-statistics and CPT theorems [16,31-33] will be discussed elsewhere.

5. Conclusion

Let us now summarize some of the results of the preceding sections.

The action (1.1) implies that a GUT-scale condensate (2.8) forms in the very early universe. It is assumed that two topological defects are “frozen into” this condensate as it forms: a cosmological instanton, which results in $U(1) \times SU(2)$ rotations of the external order parameter Ψ_{ext}, and an internal instanton, which results in rotations of the internal order parameter Ψ_{int}. Since the other fermionic and bosonic fields are born into this primordial condensate, it is natural to transform them to the frame of reference that rotates with it. In external spacetime, this leads to an action for fermions which is Lorentz-invariant at low energy (compared to an energy scale $\bar{m}$ which is presumably well above 1 TeV). The action for the initial fundamental bosons has exactly the same form as that for fermions, and is therefore quite unconventional.

Both fermions and bosons are found to have standard couplings to the gauge fields of an $SO(d)$ theory, where d is the dimension of the space containing the internal instanton. With $d = 10$, we obtain an $SO(10)$ grand-unified theory, which naturally leads to neutrino masses, coupling-constant unification, etc. It was also shown that the fermionic and bosonic fields can be quantized with either a path-integral or canonical description, even though their equations of motion are unconventional.

In this paper we did not attempt to develop a detailed phenomenological picture. However, the forms (3.49) and (3.50) imply that there are testable extensions of standard physics for fermions at high energy and for fundamental bosons which have not yet been observed. In particular, fermions have an equation of motion that violates Lorentz invariance at high energy, while the bosons discussed in the previous section violate particle-antiparticle symmetry, and other features associated with Lorentz invariance, even at low energy. For example, these bosons have spin 1/2.

References

[1] R. E. Allen, *Int. J. Mod. Phys. A* **12**, 2385 (1997) and hep-th/9612041.
[2] G. Parisi, *Field Theory, Disorder, and Simulations* (World Scientific, Singapore, 1992). Citations of the original papers are given in this and the other books and articles listed below.
[3] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, 1999).
[4] G. G. Ross, *Grand Unified Theories* (Benjamin, Menlo Park, 1984).
[5] H. Georgi, *Lie Algebras in Particle Physics* (Perseus Books, Reading, Massachusetts, 1999).
[6] P. D. B. Collins, A. D. Martin, and E. J. Squires, *Particle Physics and Cosmology* (Wiley, New York, 1989).
[7] V. D. Barger and R. J. N. Phillips, *Collider Physics* (Addison-Wesley, Reading, Massachusetts, 1997).
[8] R. N. Mohapatra, *Unification and Supersymmetry* (Springer, Berlin, 1992).
[9] H. V. Klapdor-Kleingrothaus and K. Zuber, *Particle Astrophysics* (Institute of Physics, Bristol, 2000).
[10] C. Kounnas, A. Masiero, D. V. Nanopoulos, and K. A. Olive, *Grand Unification with and without Supersymmetry and Cosmological Implications* (World Scientific, Singapore, 1984).
[11] F. Wilczek, *Nucl. Phys. Proc. Suppl.* **77**, 511 (1999) and hep-ph/9809509.
[12] R. Barbieri, L. J. Hall, S. Raby, and A. Romanino, *Nucl. Phys. B* **493**, 3 (1997) and hep-ph/9610449.
[13] P. Ramond, 1979 unpublished Caltech preprint and hep-ph/9809459.
[14] R. N. Mohapatra and P. B. Pal, *Massive Neutrinos in Physics and Astrophysics* (World Scientific, Singapore, 1998).
[15] V. Barger and K. Whisnant, in *Current Aspects of Neutrino Physics*, edited by D. O. Caldwell (Springer-Verlag, Hamburg, 2000) and hep-ph/0006235.
[16] S. Weinberg, *The Quantum Theory of Fields*, Vols. 1-3 (Cambridge University Press, Cambridge, 1995-2000). See Chapter 9 of Vol. 1 for the connection between canonical and path-integral quantization, and Chapter 5 for a proof of the spin-statistics theorem.
[17] M. E. Peskin and D. V. Schroeder, *Introduction to Quantum Field Theory* (Addison-Wesley, Reading, Massachusetts, 1995).
[18] F. Mandl and G. Shaw, *Quantum Field Theory* (Wiley, Chichester, 1993).
[19] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[20] L. H. Ryder, *Quantum Field Theory*, Second Edition (Cambridge University Press, Cambridge, 1996).
[21] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[22] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Second Edition (Clarendon, Oxford, 1993).
[23] W. Greiner and J. Reinhardt, *Field Quantization* (Springer, Berlin, 1996).
[24] P. Ramond, *Field Theory: A Modern Primer*, Second Edition (Addison-Wesley, Menlo Park, California, 1989).
[25] R. E. Allen, to be published.
[26] H. E. Haber and G. L. Kane, *Physics Reports* **117**, 75 (1985).
[27] *Perspectives on Supersymmetry*, edited by G. L. Kane (World Scientific, Singapore, 1998).
[28] *CPT and Lorentz Symmetry*, edited by V. A. Kostelecký (Singapore, World Scientific, 1999); *Proceedings of the Second Meeting on CPT and Lorentz Symmetry*, edited by V. A. Kostelecký (Singapore, World Scientific, 2002).
[29] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, Second Edition (Springer-Verlag, Heidelberg, 1976).
[30] G. Källén, *Quantum Electrodynamics* (Springer-Verlag, Heidelberg, 1972).
[31] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Princeton University Press, Princeton, 2000).
[32] A. S. Wightman, Electron. J. Diff. Eqns., Conf. 04, 2000, pp. 207-213; http://ejde.math.swt.edu. Here it is emphasized that there is no spin-statistics connection for nonrelativistic fields. I.e., a nonrelativistic field with any spin can be either bosonic or fermionic.
[33] I. Duck and E. C. G. Sudarshan, *Pauli and the Spin-Statistics Theorem* (World Scientific, Singapore, 1997). This is a historical survey with translations of excerpts from the original papers.